J Eng Math (2009) 63:51–59
DOI 10.1007/s10665-008-9245-z

A new method for the solution of convolution-type dual integral-equation systems occurring in engineering electromagnetics

F. Delfino · R. Procopio · M. Rossi

Received: 21 February 2007 / Accepted: 20 August 2008 / Published online: 12 September 2008

Abstract A general method for the solution of dual integral-equation systems is discussed in the context of applications in engineering electromagnetics. The approach follows a formulation based on a Fourier-series expansion of the unknown function and a successive expansion in a Neumann series of the Fourier-series coefficients. It is characterized by better convergence properties than classical numerical techniques usually adopted to solve the same class of problems and low computational costs. The efficiency and the performance of the proposed method are illustrated using the example of a typical electrostatic problem: the evaluation of the charge distribution of a hollow conducting cylinder, starting from the knowledge of the potential on the surface.

Keywords Convolution-type integral equations · Electrostatic problems · Fourier-series expansion

1 Introduction

A wide category of problems in engineering science can be reduced to a standard form involving dual integral equations [1,2]. Some examples can be found in the theory of elasticity, hydrodynamics, acoustics and electromagnetism, where the so-called crack, slot and strip canonical problems (see, e.g., [3, Chap. 4]) have fascinated researchers for a long time.

In many instances, the dual integral-equation system originates from the conversion of a boundary-value problem or an initial-value problem associated with a partial or an ordinary differential equation [4, Chap. 1–2], but many problems lead directly to integral equations and cannot be formulated in terms of differential equations.

Generally speaking, there is a large variety of numerical methods for the treatment of integral equations, depending on the character of the kernel, which is usually the main factor governing the choice of an appropriate quadrature formula or system of approximating functions. Various commonly occurring types of singularity call for individual treatment. Likewise, provisions can be made for cases of symmetry, periodicity or other special structures, where the solution may have special properties and economies may be effectuated in the solution process.

Integral equations arising in electromagnetism often have the form of Fredholm equations of the first kind,

F. Delfino (✉) · R. Procopio · M. Rossi
Department of Electrical Engineering, University of Genoa, Via Opera Pia 11a, 16145 Genova, Italy
e-mail: federico.delfino@unige.it

$$\int_a^b k\left(x, y, b\left(y\right)\right) \mathrm{d}y = f\left(x\right), \quad a < x < b, \tag{1.1}$$

where $k(x, y, b(y))$ is the kernel function, $b(x)$ the unknown function to be determined and $f(x)$ the free term.

If the kernel has the form

$$k(x, y, b(y)) = k(x - y)g(y, b(y)), \tag{1.2}$$

the equation is called a *convolution* integral equation and the typical solving strategy is to resort to enhanced versions of the method of moments or to numerical techniques based on the use of suitable orthogonal polynomials [5, Chap. 2], [6, Chap. 3].

Recently, a new method of solution for systems of dual integral equations has been proposed by Eswaran [7], who showed under which hypotheses the solution can exist, is unique and can be expressed as a Neumann series. Eswaran's work represents an exhaustive theory concerning dual integral equations of convolution-type and provides an effective and elegant procedure to cope with these. The main distinguishing feature of Eswaran's work is that the solution is obtained without adopting simplifying assumptions, as it typically happens in classical theories on the subject [3, Sect. 1.10], [7]. On the other hand, the major drawback of the approach, as the author himself has recognized, is its high computational cost. Actually, the method requires the construction of a matrix, whose elements are represented by integrals involving the Fourier transform of the integral-equation kernel and first-kind Bessel functions. Moreover, these integrals are highly oscillating and "slowly convergent" (quoting the author). This drawback deeply penalizes the use of Eswaran's approach for the solution of engineering or practical problems.

In the present paper, a complete reformulation of Eswaran's method will be introduced in terms of Fourier series rather than Fourier transforms and it will be demonstrated how this new approach leads to an equivalent algebraic system, where matrix coefficients are evaluated by means of summations instead of integrals. This way will enable to significantly reduce the computational costs of the algorithm without loosing accuracy of the results.

The paper is organized as follows. In Sect. 2, the developed general method of solution is presented. Then, in Sect. 3, the method is applied to the solution of an electrostatic problem and, finally, in Sect. 4, some concluding remarks are given.

2 Statement of the problem and solution of the integral-equation system in terms of Fourier series

Many engineering or physics problems frequently lead to integral equations, which can be put in the following form:

$$\begin{cases} \displaystyle\int_{-1}^{+1} f(x - y)a(y) \, \mathrm{d}y = T(x), & |x| < 1, \\ a(x) = 0, & |x| > 1, \end{cases} \tag{2.1}$$

a being the unknown function, f the kernel and T the free term.
If one defines

$$\tilde{a}(x) = \begin{cases} a(x), & |x| < 1, \\ 0, & 1 < |x| < 2, \\ \tilde{a}(x - 4), & \text{in general}, \end{cases} \tag{2.2}$$

$$\tilde{f}(x) = \begin{cases} f(x), & |x| < 2, \\ \tilde{f}(x - 4), & \text{in general}, \end{cases} \tag{2.3}$$

it can be observed that $\tilde{a}$ and $\tilde{f}$ are periodic functions.

Let us assume that they can be expanded in Fourier series, namely

$$\tilde{a}(x) = \sum_{k=-\infty}^{+\infty} A_k \mathrm{e}^{\mathrm{i}k\frac{\pi}{2}x}, \tag{2.4}$$

and

$$\tilde{f}(x) = \sum_{k=-\infty}^{+\infty} F_k \mathrm{e}^{\mathrm{i}k\frac{\pi}{2}x}, \tag{2.5}$$

where

$$A_k = \frac{1}{4}\int_{-2}^{2} \tilde{a}(x)\mathrm{e}^{-\mathrm{i}k\frac{\pi}{2}x}\,\mathrm{d}x, \tag{2.6}$$

and

$$F_k = \frac{1}{4}\int_{-2}^{2} \tilde{f}(x)\mathrm{e}^{-\mathrm{i}k\frac{\pi}{2}x}\,\mathrm{d}x. \tag{2.7}$$

With these definitions, Eq. 2.1 becomes:

$$\int_{-2}^{+2} \tilde{f}(x-y)\tilde{a}(y)\,\mathrm{d}y = T(x), \quad |x| < 1. \tag{2.8}$$

Applying the convolution theorem to the L.H.S. of (2.8), one can write:

$$\int_{-2}^{+2} \tilde{f}(x-y)\tilde{a}(y)\,\mathrm{d}y = 4\sum_{k=-\infty}^{+\infty} A_k F_k \mathrm{e}^{\mathrm{i}k\frac{\pi}{2}x}. \tag{2.9}$$

As a consequence, Eq. 2.8 becomes:

$$4\sum_{k=-\infty}^{+\infty} A_k F_k \mathrm{e}^{\mathrm{i}k\frac{\pi}{2}x} = T(x), \quad |x| < 1. \tag{2.10}$$

Now, let us assume that

$$F_k \neq 0, \quad \forall k, \tag{2.11}$$

and that, for some fixed real number I

$$\lim_k \frac{F_k}{k^I} = \ell \tag{2.12}$$

with

$$\ell \in \mathbb{R}, \quad \ell \neq 0. \tag{2.13}$$

Choosing an integer p such that $p-1 \leq I \leq p$, A_k can be written in the following form

$$A_k = \sum_{n=0}^{\infty} s_n \frac{J_{n+p}\left(\frac{k\pi}{2}\right)}{\left(\frac{k\pi}{2}\right)^p}, \tag{2.14}$$

where J is the Bessel function of the first kind and s_{n} are coefficients which must be determined. This can be achieved by simply recalling that in [7] it is shown that any function A whose Fourier transform a is of compact support can be expanded in a Neumann series, namely:

$$A(u) = \sum_{n=0}^{\infty} s_n \frac{J_{n+p}(u)}{u^p}. \tag{2.15}$$

Now, combining the definition of $A(u)$ with the second of (2.1), one can obtain:

$$A(u) = \int_{-1}^{1} a(x)\mathrm{e}^{-\mathrm{i}ux}\mathrm{d}x. \tag{2.16}$$

On the other hand, inserting the definition of $\tilde{a}$ into (2.6), one has:

$$A_k = \frac{1}{4}\int_{-1}^{1} a(x)\mathrm{e}^{-\mathrm{i}k\frac{\pi}{2}x}\mathrm{d}x. \tag{2.17}$$

Comparing (2.16) with (2.17), it is clear that:

$$A_k = \frac{1}{4}A\left(\frac{k\pi}{2}\right), \tag{2.18}$$

which ensures the validity of (2.14).

In Appendix A it is proved how Eq. 2.14 satisfies the second of (2.2) identically. At this point, the following functions can be defined:

$$\psi_n(x) = \alpha_n^{(p)} \sum_{k=-\infty}^{\infty} F_k \frac{J_{n+p}\left(\frac{k\pi}{2}\right)}{\left(\frac{k\pi}{2}\right)^p}\mathrm{e}^{\mathrm{i}k\frac{\pi}{2}x}, \tag{2.19}$$

where $\alpha_n^{(p)}$ is a normalization constant to be determined.
Substituting (2.14) and (2.19) in (2.10), one obtains:

$$T(x) = \sum_{n=0}^{\infty} \frac{s_n}{\alpha_n^{(p)}}\psi_n(x), \tag{2.20}$$

Let us construct a set of function ϕ_m, $m = 0, 1, 2, \ldots$, that is biorthogonal to the set ψ_n

$$\int_{-1}^{1} \phi_m(x)\ \psi_n(x)\mathrm{d}x = \delta_{mn}, \tag{2.21}$$

where δ_{mn} is the Kronecker operator. Then it follows that

$$\frac{s_n}{\alpha_n^{(p)}} = \int_{-1}^{1} \phi_m(x)\ T(x)\,\mathrm{d}x. \tag{2.22}$$

Once the value of $\alpha_n^{(p)}$ is suitably chosen (it will be specified later), the value of s_n can be derived, thus enabling the problem solution.

In order to find a suitable expression for $\phi_\mathrm{m}(x)$, the function ψ_n is expanded in a series of Gegenbauer polynomials $C_m^{(p)}$:

$$\psi_n(x) = \sum_{m=0}^{\infty} B_{mn}C_m^{(p)}(x). \tag{2.23}$$

Recalling the orthogonality relation [8, Eq. 7.313], which states that

$$\int_{-1}^{1} (1-x^2)^{p-\frac{1}{2}}\ C_m^{(p)}(x)\ C_n^{(p)}(x)\mathrm{d}x = \gamma_m^{(p)}\delta_{nm}, \tag{2.24}$$

where $\gamma_m^{(p)} = \frac{\pi}{m!}\frac{2^{1-2p}}{(m+p)}\frac{\Gamma(2p+m)}{[\Gamma(p)]^2}$, one may simply derive the expression for the coefficients B_{mn} multiplying Eq. 2.23 by the quantity $(1-x^2)^{p-\frac{1}{2}}\ C_r^{(p)}(x)$ and integrating over the range $(-1, 1)$, that is to say:

$$B_{rn} = \frac{1}{\gamma_r^{(p)}}\int_{-1}^{1}(1-x^2)^{p-\frac{1}{2}}\ \psi_n(x)\ C_r^{(p)}(x)\ \mathrm{d}x. \tag{2.25}$$

Inserting (2.19) into (2.25) and using (A.2), one easily gets:

$$B_{mn} = \frac{2\pi\alpha_n^{(p)}\beta_m^{(p)}}{\gamma_m^{(p)}}\sum_{k=0}^{\infty}F_k\frac{J_{n+p}\left(\frac{k\pi}{2}\right)J_{m+p}\left(\frac{k\pi}{2}\right)}{\left(\frac{k\pi}{2}\right)^{2p}}, \tag{2.26}$$

where $\beta_m^{(p)} = \left(\frac{1}{2}\right)^p\frac{i^m}{m!}\frac{\Gamma(2p+m)}{\Gamma(p)}$.

Upon assigning $\alpha_n^{(p)} = \frac{\beta_n^{(p)}}{\gamma_n^{(p)}}$, the matrix B with entries B_{mn} becomes symmetric. At this point, defining a matrix G with entries G_{ms} such that

$$\phi_m(x) = \sum_{s=0}^{\infty}\frac{1}{\gamma_s^{(p)}}G_{ms}(1-x^2)^{p-\frac{1}{2}}C_s^{(p)}(x), \tag{2.27}$$

we can impose the requirement (2.21), which means that the following condition must hold:

$$G = B^{-1}. \tag{2.28}$$

In Appendix B it is shown that B^{-1} always exists.

The expression for G can now be inserted into (2.27) to build up the set ϕ_m; then, using (2.22), the coefficients s_n can be found and finally, using (A.2), which states that $\frac{\pi\beta_n^{(p)}}{2}\frac{J_{n+p}\left(\frac{k\pi}{2}\right)^p}{\left(\frac{k\pi}{2}\right)^p}$ is the Fourier coefficient of $\left(1-x^2\right)^{p-1/2}\ \hat{C}_n^{(p)}(-x)$, one obtains:

$$\tilde{a}(x) = \sum_{n=0}^{\infty}\frac{2s_n}{\pi\beta_n^{(p)}}\hat{C}_n^{(p)}(-x)(1-x^2)^{p-1/2} \tag{2.29}$$

3 Application: evaluation of the charge distribution on the surface of a hollow conducting cylinder

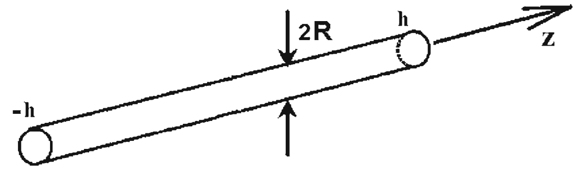
In this section, the proposed method is applied to the electrostatic problem of determining the charge distribution of a hollow conducting cylinder, once the potential V is known along its surface. In engineering electromagnetics, the solution of this problem represents the first step of the procedure leading to the evaluation of the cylinder capacitance, which in turn is useful for the calculation of the capacitance matrix of a system of multiconductor transmission lines (MTL) of finite length [9].

Let us consider the situation sketched in Fig. 1, where a perfectly conducting cylindrical tube with negligible thickness is surrounded by a dielectric medium of constant ε; the cylinder does not have end-caps. The axis of the cylinder lies along the z-axis of a system of cylindrical coordinates *(r,φ,z)*, whose origin is at the center of the tube. The radius of the cylinder is R and the height is $2h$. The surface of the cylinder, defined by $r = R$ and $-h < z < h$, has potential V.

It is well-known [10, Chap. 3], [11, Chap. 3] that, if all the charges ρ are contained in a finite volume τ, the potential can be assumed zero at infinity and can be written in the form:

$$V(P) = \frac{1}{4\pi\varepsilon}\iiint_{\tau}\frac{\rho(P_0)}{|P-P_0|}\,\mathrm{d}\tau, \tag{3.1}$$

Fig. 1 Geometry of the model problem



where $P(r, \phi, z)$ is the observation point and $P_0(r_0, \phi_0, z_0)$ the source point. The integration of (3.1) has to be performed over all the sources of the potential field, and this requirement makes it impossible to obtain a direct solution of the problem, since the function ρ is not known. However, due to the assumptions made, the charge distribution of (3.1) degenerates into one on the surface, $\sigma(z)$, and (3.1) becomes:

$$V(r,z) = \frac{R}{4\pi\varepsilon}\int_{-h}^{h}\sigma(z_0)\left[\int_{-\pi}^{\pi}\frac{\mathrm{d}(\phi-\phi_0)}{\sqrt{r^2+R^2-2rR\cos(\phi-\phi_0)+(z-z_0)^2}}\right]\mathrm{d}z_0. \tag{3.2}$$

The azimuthal symmetry implies that only the difference $\Phi = \phi - \phi_0$ is of any significance in (3.2). In order to obtain the integral equation to be solved in the unknown σ, one can place the point P on the cylinder surface, where the potential is known, thus having:

$$V(R,z) = V = \frac{R}{2\varepsilon}\int_{-h}^{h}\sigma(z_0)g(z-z_0)\mathrm{d}z_0, \tag{3.3}$$

the kernel g being defined as:

$$g(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\mathrm{d}(\Phi)}{\sqrt{4R^2\sin^2\left(\frac{\Phi}{2}\right)+x^2}}. \tag{3.4}$$

Defining now $x = z/h$, $y = z_0/h$, $f(x) = g(hx)$ and $a(x) = \sigma(hx)$, we can normalize (3.4) and rewrite it in the form (2.1), where the term $T(x)$ is the constant V, namely:

$$V = \frac{Rh}{2\varepsilon}\int_{-1}^{1}a(y)f(x-y)\,\mathrm{d}y. \tag{3.5}$$

Furthermore, it is clear that $a = 0$ for $|x| > 1$. The kernel f is an improper integral that can be rewritten as a complete elliptic integral of the first kind [8, Chap. 6]; it exhibits a logarithmic singularity for $x = 0$, namely

$$f(x) = -\frac{1}{\pi R}\log\left(\frac{|x|}{8R}\right) + b(x), \tag{3.6}$$

where b is regular in a neighborhood of the origin.

As can be seen, the kernel is square-integrable and so expansion (2.6) can be carried out. As far as the free term is concerned, its Fourier expansion is possible, since it is constant.

Finally, with reference to (2.14), $p = 0$ can be chosen. This choice satisfies the behaviour of the charge density at the wire ends [12]. In Fig. 2 the charge density is plotted as a function of the non-dimensional parameter z/h for some values of the ratio R/h. The same results can be obtained with Eswaran's method; the main difference is in the CPU time required by the two approaches. As a matter of fact, our method is less time-consuming (about 100 times faster), since it does not require the evaluation of improper integrals of the highly oscillating Bessel functions [7]. It should be noted that only five terms of the summation (2.29) are needed to get the final value.

Finally, the evaluation of the charge distribution on the cylinder surface allows us to determine the capacitance of the cylinder, which is defined as [11, Chap. 4]:

$$C = \frac{2\pi R}{V}\int_{-h}^{h}\sigma(z)\,\mathrm{d}z = \frac{2\pi R}{V}\int_{-h}^{h}\sigma(z)\,\mathrm{d}z = \frac{2\pi Rh}{V}\int_{-1}^{1}\tilde{a}\left(\frac{z}{h}\right)\mathrm{d}z. \tag{3.7}$$

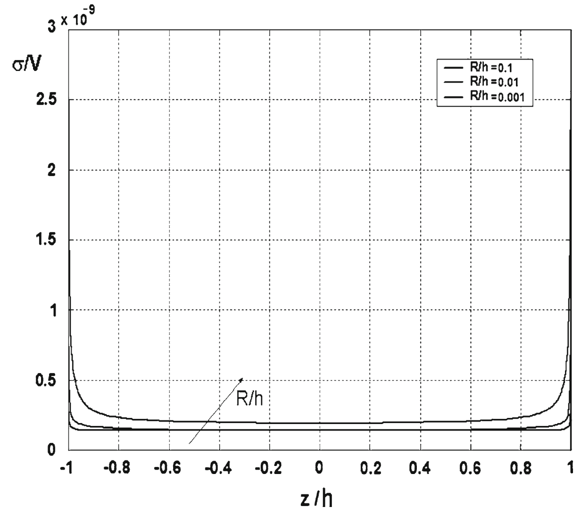


Fig. 2 Charge density on the hollow cylinder surface

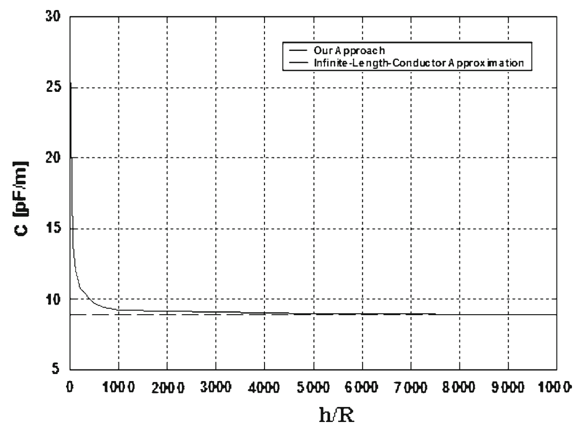


Fig. 3 Capacitance of the hollow cylinder

Recalling (2.29) and the orthogonality of the Gegenbauer polynomials, one can conclude that only the first term of the summation (2.29) is required. The value of the capacitance as a function of the ratio h/R is plotted in Fig. 3 and compared with the classical expression [11, Chap. 4] evaluated in the case of an infinitely long cylinder, with good agreement between the two approaches, as the quantity h/R increases.

4 Conclusions and perspectives of future work

An effective method to solve a class of dual integral equations has been presented in this paper. It has been shown that it consists of a complete reformulation in terms of Fourier series rather than Fourier transforms of the dual integral-equation theory developed by Eswaran [7]. The method has been applied to a relevant problem in the context of engineering electromagnetics: the evaluation of the charge distribution on the surface of a hollow conducting cylinder and the consequent calculation of the capacitance of the same cylinder. Numerical results have demonstrated that the method is reliable and characterized by fast performance.

Work is in progress in order to extend the mathematical proof to the case of integro–differential equations that actually are not completely covered by the expounded theory. Finally, further investigations will be carried out to apply the developed method to equations describing other electromagnetic phenomena.

Appendix A

The purpose of this section is to show that expansion (2.14) satisfies the second of (2.2) identically, that it is equivalent to write:

$$\tilde{a}(x) = \sum_{n=0}^{\infty} S_n \sum_{k=-\infty}^{+\infty} \frac{J_{n+p}\left(\frac{k\pi}{2}\right)}{\left(\frac{k\pi}{2}\right)^p} \mathrm{e}^{\mathrm{i}k\frac{\pi}{2}x} = 0, \quad 1 < |x| < 2. \tag{A.1}$$

If one observes that [8, Eq. 7.321]

$$\int_{-2}^{2} (1-x^2)^{p-1/2} \hat{C}_n^{(p)}(-x) \mathrm{e}^{-\mathrm{i}\frac{k\pi}{2}x} \mathrm{d}x = 2\pi \beta_n^{(p)} \frac{J_{n+p}\left(\frac{k\pi}{2}\right)^p}{\left(\frac{k\pi}{2}\right)^p}, \tag{A.2}$$

having defined $\beta_n^{(p)} = \left(\frac{1}{2}\right)^p \frac{i^n}{n!} \frac{\Gamma(2p+n)}{\Gamma(p)}$ and $\hat{C}_m^{(p)}(x)$ as

$$\hat{C}_m^{(p)}(x) = \begin{cases} C_m^{(p)}(x), & |x| < 1, \\ 0, & 1 < |x| < 2, \end{cases} \tag{A.3}$$

where $C_m^{(p)}(x)$ is the Gegenbauer polynomial of mth order, one can state that

$$\sum_{k=-\infty}^{+\infty} \frac{J_{n+p}\left(\frac{k\pi}{2}\right)}{\left(\frac{k\pi}{2}\right)^p} \mathrm{e}^{\mathrm{i}\frac{k\pi}{2}} = \frac{4}{2\pi\beta_n^{(p)}} (1-x^2)^{p-\frac{1}{2}} \hat{C}_n^{(p)}(-x) = 0 \quad \text{if } 1 < |x| < 2, \tag{A.4}$$

and conclude the proof.

Appendix B

In order to show that B^{-1} always exists, one must prove that the functions defined in (2.19) are linearly independent. Recalling (A.2) and applying the convolution theorem to (2.19), one easily gets:

$$\begin{aligned} \psi_n(x) &= \frac{\alpha_n^{(p)}}{2\pi\beta_n^{(p)}} \int_{-2}^{2} \tilde{f}(x-y)\, (1-y^2)^{p-\frac{1}{2}} \hat{C}_n^{(p)}(-y)\, \mathrm{d}y \\ &= \frac{\alpha_n^{(p)}}{2\pi\beta_n^{(p)}} \int_{-1}^{1} \tilde{f}(x-y)\, (1-y^2)^{p-\frac{1}{2}} C_n^{(p)}(-y)\, \mathrm{d}y. \end{aligned} \tag{B.1}$$

Furthermore, by defining

$$\tilde{C}_m^{(p)}(x) = \begin{cases} C_m^{(p)}(x) & |x| < 1, \\ 0, & |x| > 1, \end{cases} \tag{B.2}$$

the interval of integration in (B.1) can be extended to $(-\infty, +\infty)$ without affecting the result:

$$\psi_n(x) = \frac{\alpha_n^{(p)}}{2\pi\beta_n^{(p)}} \int_{-\infty}^{+\infty} f(x-y)(1-y^2)^{p-\frac{1}{2}} \tilde{C}_n^{(p)}(-y)\, \mathrm{d}y. \tag{B.3}$$

Now, recalling that [8, Chap. 5]:

$$\int_{-\infty}^{\infty} (1-x^2)^{p-1/2} \tilde{C}_n^{(p)}(-x) \mathrm{e}^{-\mathrm{i}ux} \mathrm{d}x = 2\pi\beta_n^{(p)} \frac{J_{n+p}(u)^p}{(u)^p}. \tag{B.4}$$

and that

$$F(u) = \int_{-\infty}^{+\infty} f(x)\, \mathrm{e}^{-\mathrm{i}ux} \mathrm{d}x, \tag{B.5}$$

one can apply the convolution theorem to (B.3) and get:

$$\psi_n(x) = \frac{\alpha_n^{(p)}}{2\pi} \int_{-\infty}^{+\infty} F(u) \frac{J_{n+p}(u)}{u^{n+p}} \mathrm{e}^{\mathrm{i}ux} \mathrm{d}u. \tag{B.6}$$

It is shown in [7] that the following functions are linearly independent:

$$h_n(x) = \alpha_n^{(p)} \int_{-\infty}^{+\infty} F(u) \frac{J_{n+p}(u)}{u^{n+p}} \mathrm{e}^{\mathrm{i}ux} \mathrm{d}u. \tag{B.7}$$

Therefore, by comparing (B.6) and (B.7), the linear independence of the functions ψ_n is proved.

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